THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Assignment 2
Due Date: 23 Feb, 2017

Recall the axioms of incidence and Playfair's axiom:

I1. For any distinct points A, B, there exists a unique line l_{AB} containing A, B.

I2. Every line contains at least two points.

I3. There exist three noncollinear points.

P. For each point A and line l, there exists unique line m such that m passes through A and m is parallel to l.

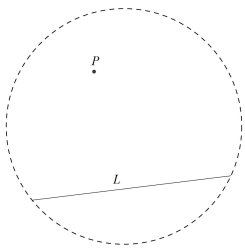
A geometry with points and lines is said to be an incidence geometry if it satisfies **I1**, **I2** and **I3**.

1. A geometry is defined as the following:
   - define a point of this geometry as any point on the unit circle centered at the origin of $\mathbb{R}^2$;
   - the unit circle itself is the only line of the geometry.

   Is this geometry an incidence geometry? Why?

2. For an incidence geometry, prove that
   (a) For any line l, there exists at least one point P that does not lie on l.
   (b) For any point P, there exists at least two distinct lines that contain P.
   (Hint: By axiom **I3**, there exist three noncollinear points R, S and T. Divide the proof into two cases that $P \in \{R, S, T\}$ and $P \notin \{R, S, T\}$.)
   (c) For any line l, there exists two distinct lines m and n that intersect l.
   (Hint: Use the result in (a).)

3. The *Klein Disk* is a geometry defined as the following:
   (a) define a point as any point in the interior of the unit disk centered at the origin of $\mathbb{R}^2$;
   (b) define a line as the intersection of any straight line in $\mathbb{R}^2$ and the Klein Disk.

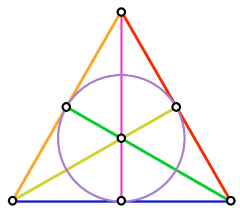


Show that the Klein Disk is an incidence geometry but not satisfies Euclid's fifth postulate.

4. An *affine plane* is an incidence geometry that satisfies the following stronger form of Playfair's axiom.

   P'. For all line l and for all point A, there exists a unique line m containing A and parallel to l.

   The diagram below shows a geometric model called Fano plane.

   Any line segment and circle shown in the diagram are considered as a line in the Fano plane. Show that the Fano plane does not satisfy **P**' and hence it is not an affine plane.

   (Remark: However, the Fano plane is an incidence geometry which satisfies **P**.)

5. A *projective plane* is a set of points and subsets called lines that satisfy the following four axioms:

PP1. Any two distinct points lie on a unique line.

PP2. Any two lines intersects at least once.

PP3. Every line contains at least three points.

PP4. There exist three noncollinear points.

   Show that the above axioms are independent.

   (Hint: Construct four geometries $(\mathcal{S}_i, \mathcal{L}_i)$, $i = 1, 2, 3, 4$, where $\mathcal{S}_i$ and $\mathcal{L}_i$ are the set of points and lines of the geometry respectively. Also, each $(\mathcal{S}_i, \mathcal{L}_i)$ does not satisfy the $i-th$ axiom but satisfy all the other.)

   (Remark: The Fano plane is a projective plane.)